TWO-FREQUENCY RADIATIVE TRANSFER AND ASYMPTOTIC SOLUTION

ALBERT C. FANNJIANG

Abstract. Two-frequency radiative transfer (RT) theory is developed for classical waves in random media. Depending on the ratio of the wavelength to the scale of medium fluctuation the two-frequency transport equation is either a Boltzmann-like integral equation or a Fokker-Planck-like differential equation in the phase space. The two-frequency transport equation is used to estimate three physical parameters: the spatial spread, the coherence length and the Thouless frequency. A closed form solution is given in the paraxial regime of geometrical radiative transfer (GRT) and shows highly nontrivial dependence of mutual coherence on the spatial displacement and frequency difference.

1. Introduction

Let $U_j, j = 1, 2$, be the random, scalar wave field of wavenumber $k_j, j = 1, 2$, The mutual coherence function and its cross-spectral version, known as the two-frequency mutual coherence function, defined by

$$\Gamma_{12}(\mathbf{x},\mathbf{y}) = \left\langle U_1(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1})U_2^*(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2})\right\rangle, \tag{1}$$

where $\langle\cdot\rangle$ stands for the ensemble averaging, is the central quantity of optical coherence theory and plays a fundamental role in analyzing propagation of random pulses [3, 14, 15]. The motivation for the scaling factors in (1) will be given below, cf. (3).

In this paper, we set out to analyze the two-frequency mutual coherence as function of the spatial displacement and frequency difference for classical waves in multiply scattering media. This problem has been extensively studied in the physics literature (see [2, 14, 19, 22, 23] and references therein). Here we derive from the multscale expansion the two-frequency version of the radiative transfer equation which is then used to estimate qualitaviely the three physical parameters: the spatial and spatial frequency spreads, and the coherence bandwidth, also known as the Thouless frequency. Moreover, we show that the two-frequency radiative transfer equation is analytically solvable in the paraxial regime of geometrical radiative transfer (GRT). The closed form solution (37) provides detailed information of the two-frequency mutual coherence beyond the current physical picture [19, 23].

To this end, we introduce the two-frequency Wigner distribution which is equivalent to the two-frequency mutual coherence and is a natural extension of the standard Wigner distribution widely used in optics [11]. A different version of two-frequency Wigner distribution for parabolic waves was introduced earlier [7] and with it the corresponding radiative transfer equation has been derived with full mathematical rigor [10]. The rigorous two-frequency radiative transfer theory for parabolic waves turns out to be the paraxial approximation of

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the theory for classical waves developed here, lending further support to the validity of the latter.

The main difference between the two-frequency radiative transfer and the standard theory is that the former retains the wave coherence and is not just transport of wave energy density.

2. Two-frequency Wigner distribution

Let $U_j, j=1,2$ be governed by the reduced wave equation

$$\Delta U_j(\mathbf{r})+k_j^2\big(\mu_j+V_j(\mathbf{r})\big)U_j(\mathbf{r})=f_j(\mathbf{r}),\quad \mathbf{r}\in\mathbb{R}^3,\quad j=1,2 \tag{2}$$

where f_j is the source term, μ_j and V_j are respectively the mean and fluctuation of the refractive index associated with the wavenumber k_j and are in general complex-valued. The imaginary part of V_j (i.e. the fluctuation of absorption coefficient) is related to its real part through the Kramer-Krönig relation. Here and below the wave speed is set to be unity. To solve (2) one needs also some boundary condition which is assumed to be at the far field (i.e. $|\mathbf{x}|=\infty$).

We introduce the two-frequency Wigner distribution

$$W(\mathbf{x},\mathbf{p})=\frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1})U_2^*(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2})d\mathbf{y} \tag{3}$$

where the choice of the scaling factors is crucial. In other words, the ensemble average $\langle W\rangle$ is just the (partial) Fourier transform of the mutual coherence function (1). In view of the definition, we see that both $\mathbf{x}$ and $\mathbf{p}$ are dimensionless. The two-frequency Wigner distribution defined here has a different scaling from the one introduced for the parabolic waves [7].

The purpose of introducing the two-frequency Wigner distribution is to develop a two-frequency theory in analogy to the well studied standard theory of radiative transfer. Although the definition (3) requires the domain to be $\mathbb{R}^d$, the governing equation for $\langle W\rangle$, once obtained, can be (inverse) Fourier transformed back to get the governing equation for Γ_{12} for which the boundary condition at *finite* boundaries is more convenient to describe.

The Wigner distribution has the following easy-to-check properties:

$$\int |W|^2(\mathbf{x},\mathbf{p})d\mathbf{x}d\mathbf{p} = \left(\frac{\sqrt{k_1k_2}}{2\pi}\right)^3\int |U_1|^2(\mathbf{x})d\mathbf{x}\int |U_2|^2(\mathbf{x})d\mathbf{x}$$

$$\int W(\mathbf{x},\mathbf{p})e^{i\mathbf{p}\cdot\mathbf{y}}d\mathbf{p} = U_1(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1})U_2^*(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2}) \tag{4}$$

$$\int W(\mathbf{x},\mathbf{p})e^{-i\mathbf{x}\cdot\mathbf{q}}d\mathbf{x} = \left(\pi^2k_1k_2\right)^3\widehat{U}_1(\frac{k_1\mathbf{p}}{4}+\frac{k_1\mathbf{q}}{2})\widehat{U}_2^*(\frac{k_2\mathbf{p}}{4}-\frac{k_2\mathbf{q}}{2}), \tag{5}$$

where $\widehat{\cdot}$ stands for the Fourier transform, and hence contains all the information in the two-point two-frequency function. In particular,

$$\int \mathbf{p}W(\mathbf{x},\mathbf{p})d\mathbf{p} = -i\Big[\frac{1}{2k_1}\nabla U_1(\frac{\mathbf{x}}{k_1})U_2^*(\frac{\mathbf{x}}{k_2})-\frac{1}{2k_2}U_1(\frac{\mathbf{x}}{k_1})\nabla U_2^*(\frac{\mathbf{x}}{k_2})\Big]$$

which, in the case of $k_1=k_2$, is proportional to the energy flux density.

We now derive the equation for the two-frequency Wigner distribution. After taking the derivative $\mathbf{p}\cdot\nabla$ and some calculation we have

$$
\begin{aligned}
\mathbf{p}\cdot\nabla W &= \frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1})U_2^*(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2})V_1(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1})d\mathbf{y}\\
&\quad -\frac{i}{2(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1(\frac{\mathbf{x}}{k_1}+\frac{\mathbf{y}}{2k_1})U_2^*(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2})V_2^*(\frac{\mathbf{x}}{k_2}-\frac{\mathbf{y}}{2k_2})d\mathbf{y}\\
&\quad +\frac{i}{2}(\mu_1-\mu_2^*)W+F
\end{aligned}
$$

where the function F depends linearly on U_1, U_2.

Substituting the spectral representation of V_j

$$
V_j(\mathbf{x}) = \int e^{-i\mathbf{q}\cdot\mathbf{x}}\hat{V}_j(d\mathbf{q}) \tag{6}
$$

in the expression and using the definition of W we then obtain the exact equation

$$
\begin{aligned}
&\mathbf{p}\cdot\nabla W-\frac{i}{2}(\mu_1-\mu_2^*)W-F\\
&= \frac{i}{2}\int \hat{V}_1(d\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}/k_1}W(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1})-\frac{i}{2}\int \hat{V}_2^*(d\mathbf{q})e^{-i\mathbf{q}\cdot\mathbf{x}/k_2}W(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_2}).
\end{aligned} \tag{7}
$$

Here and below $\hat{V}_2^*$ is the complex-conjugate of the Fourier spectral measure $\hat{V}_2$.

Let us pause to compare the classical wave with the quantum wave function in the context of two-frequency formulation. The quantum wave functions Ψ_j at two different frequencies ω_1, ω_2 satisfy the stationary Schrödigner equaiton

$$
\frac{\hbar^2}{2}\Delta\Psi_j+\big(\mu+V(\mathbf{x})\big)\Psi_j = -\omega_j\hbar\Psi_j,\quad j=1,2, \tag{8}
$$

where μ and V are real-valued. The natural definition of the two-frequency Wigner distribution for the quantum wave functions is

$$
W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}\Psi_1(\mathbf{x}+\frac{\hbar\mathbf{y}}{2})\Psi_2^*(\mathbf{x}-\frac{\hbar\mathbf{y}}{2})d\mathbf{y} \tag{9}
$$

which satisfies the Wigner-Moyal equation

$$
\mathbf{p}\cdot\nabla W+i(\omega_1-\omega_2)W = \frac{i}{\hbar}\int \hat{V}(d\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}}\Big[W(\mathbf{x},\mathbf{p}-\frac{\hbar\mathbf{q}}{2})-W(\mathbf{x},\mathbf{p}+\frac{\hbar\mathbf{q}}{2})\Big]. \tag{10}
$$

3. Two-frequency radiative transfer scaling

We assume that $V_j(\mathbf{x}), j=1,2$ is a centered statistically homogeneous random field admitting the spectral representation (6) with the spectral measure $\hat{V}_j(\cdot)$ satisfying

$$
\mathbb{E}[\hat{V}_j(d\mathbf{p})\hat{V}_j(d\mathbf{q})] = \delta(\mathbf{p}+\mathbf{q})\Phi_j(\mathbf{p})d\mathbf{p}d\mathbf{q},\quad j=1,2
$$

where Φ_j is the power spectrum of the random field. The above δ function is a consequence of the statistical homogeneity of the random field V_j.

When V_j is real-valued, the power spectral density $\Phi_j(\mathbf{p})$ is real-valued, non-negative and satisfies $\Phi_j(\mathbf{p}) = \Phi_j(-\mathbf{p}), \forall\mathbf{p}$. We will also need the cross-frequency correlation and we postulate the existence of the cross-frequency spectrum Φ_{12} such that

$$
\mathbb{E}[\hat{V}_1(d\mathbf{p})\hat{V}_2(d\mathbf{q})] = \delta(\mathbf{p}+\mathbf{q})\Phi_{12}(\mathbf{p})d\mathbf{p}d\mathbf{q}.
$$

An important regime of multiple scattering of classical waves takes place when the scale of medium fluctuation is much smaller than the propagation distance but is comparable or much larger than the wavelength [14, 16]. Radiative transfer regime is equivalent to the scaling limit which replaces $\mu_j + V_j$ in eq. (2) with

$$\frac{1}{\gamma^2\varepsilon^2}\Big(\mu_j + \sqrt{\varepsilon}V_j(\frac{\mathbf{x}}{\varepsilon})\Big), \quad \gamma > 0, \quad \varepsilon \ll 1 \tag{11}$$

where ε is the ratio of the scale of medium fluctuation to the $O(1)$ propagation distance. The prefactor $(\gamma\varepsilon)^{-2}$ arises from rescaling the wavenumber as $k \to k/(\varepsilon\gamma)$ with $\gamma = O(1)$ being the ratio of the wavelength to the scale of medium fluctuation. The resulting medium fluctuation

$$\frac{1}{\varepsilon^{3/2}}V\Big(\frac{\mathbf{x}}{\varepsilon}\Big)$$

converges to a spatial white-noise in three dimensions.

Physically the radiative transfer scaling belongs to the diffusive wave regime under the condition of a large dimensionless conductance $g = N\ell_t/L$, where ℓ_t is the transport mean free path, L is the sample size in the direction of propagation and $N = 2\pi A/\lambda^2$ is the number of transverse modes, limited by the illuminated area A and the wavelength of radiation λ, [2, 19]. The dimensionless conductance g can be expressed as $g = k\ell_t\theta$ with the Fresnel number $\theta = A/(\lambda L)$. With the scaling (11), $k\ell_t \sim \theta \sim \gamma^{-1}\varepsilon^{-1}$ and hence $g \sim \gamma^{-2}\varepsilon^{-2} \gg 1$ for any finite γ as $\varepsilon \to 0$.

Anticipating small-scale fluctuation in the mutual coherence we modify the definition of the two-frequency Wigner distribution in the following way

$$W(\mathbf{x},\mathbf{p}) = \frac{1}{(2\pi)^3}\int e^{-i\mathbf{p}\cdot\mathbf{y}}U_1(\frac{\mathbf{x}}{k_1} + \frac{\gamma\varepsilon\mathbf{y}}{2k_1})U_2^*(\frac{\mathbf{x}}{k_2} - \frac{\gamma\varepsilon\mathbf{y}}{2k_2})d\mathbf{y}$$

Eq. (7) now becomes

$$\mathbf{p}\cdot\nabla W - F \quad = \quad \frac{i}{2\varepsilon\gamma}(\mu_1 - \mu_2^*)W + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}W \tag{12}$$

where the operator $\mathcal{L}$ is defined by

$$\mathcal{L}W(\mathbf{x},\mathbf{p}) \quad = \quad \frac{i}{2\gamma}\int \hat{V}_1(d\mathbf{q})e^{i\frac{\mathbf{q}\cdot\mathbf{x}}{\varepsilon k_1}}W(\mathbf{x},\mathbf{p} - \frac{\gamma\mathbf{q}}{2k_1}) - \frac{i}{2\gamma}\int \hat{V}_2^*(d\mathbf{q})e^{-i\frac{\mathbf{q}\cdot\mathbf{x}}{\varepsilon k_2}}W(\mathbf{x},\mathbf{p} - \frac{\gamma\mathbf{q}}{2k_2}).$$

To capture the cross-frequency correlation in the radiative transfer regime we also need to restrict the frequency difference range

$$\lim_{\varepsilon\to 0} k_1 = \lim_{\varepsilon\to 0} k_2 = k, \quad \lim_{\varepsilon\to 0}\varepsilon^{-1}\gamma^{-1}k^{-1}(k_2 - k_1) = \beta > 0. \tag{13}$$

Here the normalized wavenumber k should be considered as close to unity. Assuming the differentiability of the mean refractive index's dependence on the wavenumber we write

$$\lim_{\varepsilon\to 0}\frac{\mu_2^* - \mu_1}{2\varepsilon\gamma} = \mu'. \tag{14}$$

3.1. **Two-frequency geometrical optics.** Before we proceed, let us pause to discuss the geometrical optics limit which will bring (12) into a more familiar form. The geometrical optics regime corresponds to the vanishing ratio γ of the wavelength to the scale of medium fluctuation while $\varepsilon > 0$ is held fixed.

To this end, the conditions (13) and (14) for the frequency range need to be modified analogously:

$$\lim_{\gamma\to 0} k_1 = \lim_{\gamma\to 0} k_2 \quad = \quad k \tag{15}$$

$$\lim_{\gamma\to 0} \gamma^{-1}\varepsilon^{-1}k^{-1}(k_2 - k_1) \quad = \quad \beta > 0 \tag{16}$$

$$\lim_{\gamma\to 0} (2\varepsilon\gamma)^{-1}(\mu_2^* - \mu_1) \quad = \quad \mu'. \tag{17}$$

Passing to the limit $\gamma \to 0$ in (12) we obtain the first-order partial differential equation

$$\mathbf{p}\cdot\nabla_{\mathbf{x}}W(\mathbf{x},\mathbf{p}) + i\mu' W(\mathbf{x},\mathbf{p}) - F \quad = \quad \frac{1}{2k\sqrt{\varepsilon}}(\nabla V)\Big(\frac{\mathbf{x}}{\varepsilon k}\Big)\cdot\Big[\nabla_{\mathbf{p}} - i\beta\mathbf{x}\Big]W(\mathbf{x},\mathbf{p}). \tag{18}$$

For $\beta = 0$ (then $i\mu' \sim$ the imaginary part of μ_1), eq. (18) is the static Liouville equation that governs the phase space energy density. For $\beta > 0$, eq. (18) retains the wave character and we shall refer to it as the two-frequency Liouville equation.

3.2. **Multi-scale expansion (MSE).** To derive the radiative transfer equation for the two-frequency Wigner distribution we employ the multi-scale expansion (MSE) [1, 17]. MSE begins with introducing the fast variable

$$\tilde{\mathbf{x}} = \mathbf{x}/\varepsilon$$

and treating $\tilde{\mathbf{x}}$ as independent from the slow variable $\mathbf{x}$. Consequently the derivative $\mathbf{p}\cdot\nabla$ consists of two terms

$$\mathbf{p}\cdot\nabla = \mathbf{p}\cdot\nabla_{\mathbf{x}} + \varepsilon^{-1}\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}. \tag{19}$$

Then MSE posits the following asymptotic expansion:

$$W(\mathbf{x},\mathbf{p}) = \bar{W}(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) + \sqrt{\varepsilon}W_1(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) + \varepsilon W_2(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) + O(\varepsilon^{3/2}), \quad \tilde{\mathbf{x}} = \mathbf{x}\varepsilon^{-1} \tag{20}$$

whose proper sense will be explained in the Appendix.

Substituting the ansatz into eq. (12) and using (19) we determine each term of (20) by equating terms of the same order of magnitude starting with the highest order ε^{-1}.

The ε^{-1}-order equation has one term:

$$\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}\bar{W} = 0$$

which can be solved by setting $\bar{W} = \bar{W}(\mathbf{x},\mathbf{p})$. Namely, to the leading order W is independent of the fast variable.

The next is the $\varepsilon^{-1/2}$-order equation:

$$\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_1 = \mathcal{L}\bar{W}$$

which can be solved approximately by introducing a small positive regularization parameter η as

$$W_1^\eta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}) = \frac{i}{2\gamma}\int \hat{V}_1(d\mathbf{q})\frac{e^{i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_1}}}{\eta+i\mathbf{q}\cdot\mathbf{p}/k_1}\bar{W}(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k}) - \frac{i}{2\gamma}\int \hat{V}_2^*(d\mathbf{q})\frac{e^{-i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_2}}}{\eta-i\mathbf{q}\cdot\mathbf{p}/k_2}\bar{W}(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k}). \tag{21}$$

Namely, W_1^η solves exactly the following equation

$$\eta W_1^\eta + \mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_1^\eta = \mathcal{L}\bar{W}. \tag{22}$$

We will set $\eta=\varepsilon$ for reason explicated in the Appendix.

Finally the $O(1)$ equation is

$$\begin{aligned}&\mathbf{p}\cdot\nabla_{\mathbf{x}}\bar{W}(\mathbf{x},\mathbf{p})+\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_2(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p})+i\mu'\bar{W}-F\\ &= \frac{i}{2\gamma}\int\hat{V}_1(d\mathbf{q})e^{i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_1}}W_1^\eta(\mathbf{x},\tilde{\mathbf{x}},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k})-\frac{i}{2\gamma}\int\hat{V}_2^*(d\mathbf{q})e^{-i\frac{\mathbf{q}\cdot\tilde{\mathbf{x}}}{k_2}}W_1^\eta(\mathbf{x},\tilde{\mathbf{x}}\mathbf{p}-\frac{\gamma\mathbf{q}}{2k}).\end{aligned} \tag{23}$$

The necessary condition to solve W_2 from (23) is that $\langle\mathbf{p}\cdot\nabla_{\tilde{\mathbf{x}}}W_2\rangle=0$. Hence using (21) in (23), taking the ensemble average and passing to the limit $\eta\to 0$ we obtain the governing equation for $\bar{W}$:

$$\begin{aligned}&\mathbf{p}\cdot\nabla_{\mathbf{x}}\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p})+i\mu'\left\langle\bar{W}\right\rangle-\langle F\rangle\\
&= -\frac{k_1^3}{2\gamma^4}\int d\mathbf{q}\Phi_1\big(\frac{k_1}{\gamma}(\mathbf{p}-\mathbf{q})\big)\pi\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p})+\frac{ik_1^3}{2\gamma^4}⨍ d\mathbf{q}\frac{\Phi_1\big(\frac{k_1}{\gamma}(\mathbf{p}-\mathbf{q})\big)}{|\mathbf{p}|^2-|\mathbf{q}|^2}\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p})\\
&\quad -\frac{k_2^3}{2\gamma^4}\int d\mathbf{q}\Phi_2\big(\frac{k_2}{\gamma}(\mathbf{p}-\mathbf{q})\big)\pi\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p})-\frac{ik_2^3}{2\gamma^4}⨍ d\mathbf{q}\frac{\Phi_2\big(\frac{k_2}{\gamma}(\mathbf{p}-\mathbf{q})\big)}{|\mathbf{p}|^2-|\mathbf{q}|^2}\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p})\\
&\quad +\frac{1}{4\gamma^2}\int d\mathbf{q}\Phi_{12}(\mathbf{q})e^{i\mathbf{x}\cdot\mathbf{q}(k_1^{-1}-k^{-2})}\pi\delta\big(\frac{\mathbf{q}}{k_2}\cdot(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1})\big)\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}-\frac{\gamma\mathbf{q}}{2k_2})\\
&\quad +\frac{1}{4\gamma^2}\int d\mathbf{q}\Phi_{12}(\mathbf{q})e^{i\mathbf{x}\cdot\mathbf{q}(k_1^{-1}-k^{-2})}\pi\delta\big(\frac{\mathbf{q}}{k_1}\cdot(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_2})\big)\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}-\frac{\gamma\mathbf{q}}{2k_2})\\
&\quad +\frac{i}{4\gamma^2}⨍ d\mathbf{q}\Big[\frac{1}{\frac{\mathbf{q}}{k_2}\cdot(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1})}-\frac{1}{\frac{\mathbf{q}}{k_1}\cdot(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_2})}\Big]\Phi_{12}(\mathbf{q})e^{i\mathbf{x}\cdot\mathbf{q}(k_1^{-1}-k^{-2})}\left\langle\bar{W}\right\rangle(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{2k_1}-\frac{\gamma\mathbf{q}}{2k_2})\end{aligned}$$

where $⨍$ denotes the Cauchy principal value integral and $\langle F\rangle$ depends only on the mean fields $\langle U_1\rangle, \langle U_2\rangle$ and will be treated as known throughout the paper. Here we have used the fact that in the sense of generalized function

$$\lim_{\eta\to 0}\frac{1}{\eta+i\xi}=\pi\delta(\xi)-\frac{i}{\xi}$$

with the second term giving rise to the Cauchy principal value integral.

Due to the assumption (13) we have $\lim\Phi_1=\lim\Phi_2=\lim\Phi_{12}=\Phi$ provided that the refractive index is a continuous function of the frequency. All the integrals with the imaginary number i as a prefactor drop out. With some changes of variables the governing equation for $\langle \bar{W} \rangle$ takes the much simplified form:

$$\begin{aligned} (24)\quad & \mathbf{p}\cdot\nabla_{\mathbf{x}} \langle \bar{W} \rangle + i\mu' \langle \bar{W} \rangle - \langle F \rangle \\ &= \frac{\pi k^3}{\gamma^4}\int d\mathbf{q}\Phi\big(\frac{k}{\gamma}(\mathbf{p}-\mathbf{q})\big)\delta(|\mathbf{p}|^2-|\mathbf{q}|^2)\Big[e^{i\mathbf{x}\cdot(\mathbf{p}-\mathbf{q})\beta\gamma/k}\langle \bar{W} \rangle(\mathbf{x},\mathbf{q}) - \langle \bar{W} \rangle(\mathbf{x},\mathbf{p})\Big]. \end{aligned}$$

The δ-function in the scattering kernel is a result of elastic scattering which preserve the wavenumber. When $\beta = 0$ (then $i\mu' \sim$ the imaginary part of μ_1), eq. (24) reduce to the standard form of radiative transfer equation for the phase space energy density [18, 13, 4, 16]. For $\beta > 0$, the wave featue is retained in (24). When $\beta \to \infty$, the first term on the right hand side of (24) drops out, due to rapid phase fluctuation, so the random scattering effect is pure damping.

4. Two-frequency RT for paraxial wave

When the backscattering can be neglected such as in laser beam propagation in turbulent media the paraxial approximation is widely used. Let us assume that the wave propagates mainly in the z direction. Let p and $\mathbf{p}_\perp$ denote the longitudinal and transverse components of $\mathbf{p}$, respectively. Let q and $\mathbf{q}_\perp$ be likewise defined. In the paraxial approximation, $\langle \bar{W} \rangle$ is concentrated in the region $|\mathbf{p}_\perp| \ll p \approx 1$ and $|\mathbf{q}_\perp| \ll q \approx 1$ so we can write $\langle \bar{W} \rangle = \langle \bar{W} \rangle(z, \mathbf{x}_\perp, \mathbf{p}_\perp)$ and approximate eq. (24) by

$$\begin{aligned} & \partial_z \langle \bar{W} \rangle + \mathbf{p}_\perp\cdot\nabla_{\mathbf{x}_\perp} \langle \bar{W} \rangle + i\mu' \langle \bar{W} \rangle - \langle F \rangle \\ &= \frac{\pi k^2}{\gamma^3}\int d\mathbf{q}_\perp \int dq\Phi\big(q, \frac{k}{\gamma}(\mathbf{p}_\perp-\mathbf{q}_\perp)\big)\delta(|\mathbf{p}_\perp|^2-|\mathbf{q}_\perp|^2) \\ (25)\quad & \times\Big[e^{i\mathbf{x}_\perp\cdot(\mathbf{p}_\perp-\mathbf{q}_\perp)\beta\gamma/k}\langle \bar{W} \rangle(z,\mathbf{x}_\perp,\mathbf{q}_\perp) - \langle \bar{W} \rangle(z,\mathbf{x}_\perp,\mathbf{p}_\perp)\Big] \end{aligned}$$

which agrees with the rigorous two-frequency transport theory for the paraxial wave equation [10]. The enormous simplification attained in the paraxial approximation is that only the one-sided boundary condition (e.g. at $z = -\infty$) is needed to solve eq. (25).

5. Two-frequency geometrical radiative transfer

Let us consider the further limit $\gamma \ll 1$ when the wavelength is much shorter than the correlation length of the medium fluctuation. To this end, the following form is more convenient to work with

$$\begin{aligned} (26)\quad & \mathbf{p}\cdot\nabla_{\mathbf{x}} \langle \bar{W} \rangle + i\mu' \langle \bar{W} \rangle - \langle F \rangle \\ &= \frac{\pi k}{2\gamma^2}\int d\mathbf{q}\Phi(\mathbf{q})\delta\big(\mathbf{q}\cdot(\mathbf{p}-\frac{\gamma\mathbf{q}}{2k})\big)\Big[e^{i\mathbf{x}\cdot\mathbf{q}\beta\gamma/k}\langle \bar{W} \rangle\big(\mathbf{x},\mathbf{p}-\frac{\gamma\mathbf{q}}{k}\big) - \langle \bar{W} \rangle(\mathbf{x},\mathbf{p})\Big] \end{aligned}$$

which is obtained from eq. (24) after a change of variables.

Now we make an additional assumption that $\Phi(\mathbf{q}) = \Phi(-\mathbf{q})$ which is the case for real-valued V. We expand the right hand side of (26) in γ and pass to the limit $\gamma \to 0$ to obtain

$$(27)\quad \mathbf{p}\cdot\nabla_{\mathbf{x}} \langle \bar{W} \rangle + i\mu' \langle \bar{W} \rangle - \langle F \rangle = -\frac{1}{4k}(i\nabla_{\mathbf{p}} - \beta\mathbf{x})\cdot\mathbf{D}\cdot(i\nabla_{\mathbf{p}} - \beta\mathbf{x})\langle \bar{W} \rangle$$

with the diffusion coefficient

$$\mathbf{D}(\mathbf{p}) = \pi \int \Phi(\mathbf{q})\delta(\mathbf{p}\cdot\mathbf{q})\mathbf{q}\otimes\mathbf{q}d\mathbf{q}. \tag{28}$$

It is noteworthy that eq. (27) can also be derived by starting with the two-frequency Liouville equation (18) and following the same MSE procedure.

5.1. **Spatial (frequency) spread and coherence bandwidth.** Through dimensional analysis, eq. (27) yields qualitative information about important physical parameters of the stochastic channel. To show this, let us assume for simplicity the isotropy of the medium, i.e. $\Phi(\mathbf{p}) = \Phi(|\mathbf{p}|)$, so that $\mathbf{D} = C|\mathbf{p}|^{-1}P(\mathbf{p})$ where

$$C = \frac{\pi}{3}\int \delta\Big(\frac{\mathbf{p}}{|\mathbf{p}|}\cdot\frac{\mathbf{q}}{|\mathbf{q}|}\Big)\Phi(|\mathbf{q}|)|\mathbf{q}|d\mathbf{q}$$

is a constant and $P(\mathbf{p})$ the orthogonal projection onto the plane perpendicular to $\mathbf{p}$.

Note again that the variables $\mathbf{x}$ and $\mathbf{p}$ are dimensionless. Now consider the following change of variables

$$\mathbf{x} = \sigma_x k\tilde{\mathbf{x}}, \quad \mathbf{p} = \sigma_p\tilde{\mathbf{p}}/k, \quad \beta = \beta_c\tilde{\beta} \tag{29}$$

where σ_x and σ_p are respectively the spreads in position and spatial frequency, and β_c is the coherence bandwidth, also known as the Thouless frequency. Let us substitute (29) into eq. (27) and aim for the standard form

$$\tilde{\mathbf{p}}\cdot\nabla_{\tilde{\mathbf{x}}}\left\langle \bar{W}\right\rangle + i\mu'\left\langle \bar{W}\right\rangle - \left\langle F\right\rangle = -\Big(i\nabla_{\tilde{\mathbf{p}}} - \tilde{\beta}\tilde{\mathbf{x}}\Big)\cdot|\tilde{\mathbf{p}}|^{-1}P(\mathbf{p})\Big(i\nabla_{\tilde{\mathbf{p}}} - \tilde{\beta}\tilde{\mathbf{x}}\Big)\left\langle \bar{W}\right\rangle. \tag{30}$$

The 1-st term on the left side yields the first duality relation

$$\sigma_x/\sigma_p \sim 1/k^2. \tag{31}$$

The balance of terms in each pair of parentheses yields the second duality relation

$$\sigma_x\sigma_p \sim \frac{1}{\beta_c}. \tag{32}$$

Finally the removal of the constant C determines

$$\sigma_p \sim k^{2/3}C^{1/3} \tag{33}$$

from which σ_x and β_c can be determined by using (31) and (32). In particular we obtain the scaling behavior of the Thouless frequency:

$$\beta_c \sim k^{2/3}C^{-2/3}.$$

We do not know if, as it stands, eq. (30) is analytically solvable but we can solve it analytically in the paraxial approximation discussed next.

5.2. **Paraxial approximation of GRT: exact solution.** We use here the paraxial setting and notation defined in Section 4. In the paraxial approximation, $P(\mathbf{p})$ becomes the orthogonal projection onto the transverse plane and eq. (27) reduces

$$\left[\partial_z + \mathbf{p}_\perp \cdot \nabla_{\mathbf{x}_\perp}\right] \left\langle \bar{W} \right\rangle + i\mu' \left\langle \bar{W} \right\rangle - \left\langle F \right\rangle = -\frac{C}{4k} \left(i\nabla_{\mathbf{p}_\perp} - \beta \mathbf{x}_\perp\right)^2 \left\langle \bar{W} \right\rangle. \quad (34)$$

For eq. (34) only the one-sided boundary condition (e.g. at $z = -\infty$) is needed.

Let σ_* the spatial spread in the transverse coordinates $\mathbf{x}_\perp$, ℓ_c the coherence length in the transverse dimensions and β_c the coherence bandwidth. We then seek the following change of variables

$$\tilde{\mathbf{x}}_\perp = \frac{\mathbf{x}_\perp}{\sigma_* k}, \quad \tilde{\mathbf{p}}_\perp = \mathbf{p}_\perp k \ell_c, \quad \tilde{z} = \frac{z}{Lk}, \quad \tilde{\beta} = \frac{\beta}{\beta_c}$$

to remove all the physical parameters from (27) and to aim for the form

$$\partial_{\tilde{z}} \left\langle \bar{W} \right\rangle + \tilde{\mathbf{p}}_\perp \cdot \nabla_{\tilde{\mathbf{x}}_\perp} \left\langle \bar{W} \right\rangle + Lki\mu' \left\langle \bar{W} \right\rangle - Lk \left\langle F \right\rangle = -\left(i\nabla_{\tilde{\mathbf{p}}_\perp} - \tilde{\beta}\tilde{\mathbf{x}}_\perp\right)^2 \left\langle \bar{W} \right\rangle. \quad (35)$$

The same reasoning as above now leads to

$$\ell_c \sigma_* \sim L/k, \quad \sigma_*/\ell_c \sim 1/\beta_c, \quad \ell_c \sim 1/(k\sqrt{LC})$$

and hence the following scaling behavior of the Thouless frequency

$$\beta_c \sim k^{-1} C^{-1} L^{-2}.$$

After the inverse Fourier transform eq. (35) becomes

$$\partial_{\tilde{z}} \Gamma - i \nabla_{\tilde{\mathbf{y}}_\perp} \cdot \nabla_{\tilde{\mathbf{x}}_\perp} \Gamma + Lki\mu' \Gamma - Lk \left\langle F \right\rangle = -\left|\tilde{\mathbf{y}}_\perp + \tilde{\beta}\tilde{\mathbf{x}}_\perp\right|^2 \Gamma \quad (36)$$

which is the governing equation for the two-frequency mutual coherence in the normalized variables. Eq. (36) can be solved analytically and its Green function is given by

$$\begin{aligned}
(37) \quad & \frac{(1+i)(2\tilde{\beta})^{1/2}}{(2\pi)^2 \sin\left((2\tilde{\beta})^{1/2}(1+i)\right)} \exp\left[-iLk\mu'\right] \exp\left[i\frac{|\tilde{\mathbf{y}}_\perp - \mathbf{y}'_\perp|^2}{4\tilde{\beta}}\right] \exp\left[i\frac{\tilde{\beta}|\tilde{\mathbf{x}}_\perp - \mathbf{x}'_\perp|^2}{4}\right] \\
& \times \exp\left[i\frac{(\tilde{\mathbf{y}}_\perp - \mathbf{y}'_\perp)\cdot(\tilde{\mathbf{x}}_\perp - \mathbf{x}'_\perp)}{2}\right] \exp\left[-\frac{1-i}{2(2\tilde{\beta})^{1/2}} \left|\mathbf{y}'_\perp - \tilde{\beta}\mathbf{x}'_\perp\right|^2 \tan\left((2\tilde{\beta})^{1/2}(1+i)\right)\right] \\
& \times \exp\left[\frac{1-i}{2(2\tilde{\beta})^{1/2}} \cot\left((2\tilde{\beta})^{1/2}(1+i)\right) \left|\tilde{\mathbf{y}}_\perp - \tilde{\beta}\tilde{\mathbf{x}}_\perp - \frac{\mathbf{y}'_\perp - \tilde{\beta}\mathbf{x}'_\perp}{\cos\left((2\tilde{\beta})^{1/2}(1+i)\right)}\right|^2\right]
\end{aligned}$$

for $\tilde{z} = 1$. The apparent singular nature of the limit $\tilde{\beta} \to 0$ in (37) is deceptive. Indeed, the vanishing-$\tilde{\beta}$ limit is regular and converges to the Green function of eq. (36) with $\tilde{\beta} = 0$.

Formula (37) is consistent with the current result in the literature [19, 23]

$$\sim \exp\left[-\sqrt{2\tilde{\beta}}\right]$$

which is just the large $\tilde{\beta}$ asymptotic of the factor $1/\sin\left((2\tilde{\beta})^{1/2}(1+i)\right)$. Moreover (37) provides detailed information about the simultaneous dependence of the mutual coherence on the frequency difference and spatial displacement. A closely related equation arises in the two-frequency formulation of the Markovian approximation of the paraxial waves [7]. The closed form solution is crucial for analyzing the performance of time reversal communication with broadband signals [9]. The solution procedure for (37) is similar to that given in [9] and is omitted here.

6. Discussion and conclusion

At least two approaches to the derivation of the standard (one-frequency) radiative transfer equation from the wave equation exist in the literature: the diagrammatic expansion method [22, 16] and the multi-scale expansion method advocated here [1]. The latter is considerably simpler than the former in terms of the amount of calculation involved. Both approaches have been developed with full mathematical rigor in some special cases (see [8] and the references therein). With the framework of the multi-scale expansion the two-frequency radiative transfer equation can be rigorously derived for the one-sided boundary value problem of the paraxial wave equation by using the so called martingale method in probability theory [10]. The martingale method is unfortunately not applicable to the general (two-sided) boundary value problem such as (2).

On the other hand, the two-frequency radiative transfer limit for (2) may be dealt with by extending the diagrammatic method developed for the time dependent Schrödinger equation [20, 21, 12, 5]. By analogy, (2) is like the stationary Schrödinger equation with an energy-dependent potential. However, the diagrammatic approach, rigorous or not, is more complicated than MSE to carry out, so, in addition to the formal expansion, we will be content to give a brief analysis of MSE in the Appendix.

Using MSE we have given a formal derivation of the two-frequency radiative transfer equation for the classical wave equation in terms of the new two-frequency Wigner distribution. The validity of the derivation is supported by the interchangeability of the paraxial approximation and the two-frequency radiative transfer limit. A main feature of the two-frequency radiative transfer equation is that it retains some wave character.

By dimensional analysis with the two-frequency radiative transfer equation, we obtain the qualitative behavior of the spatial spread, the spatial frequency spread and the coherent bandwidth. In the paraxial regime of GRT, we obtain a closed form solution, revealing highly non-trivial structure of the two-frequency mutual coherence function.

The present approach can be generalized to the polarized waves described by the Maxwell equations or the vector wave equation, which will be presented elsewhere.

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Appendix A. Meaning of MSE and the regularization parameter

Let us determine what the reasonable choice of the regularization parameter η should be. First note that $\lim_{\eta\to 0}\eta\int\langle|W_1^\eta|^2\rangle\, d\mathbf{x}d\mathbf{p} > 0$ in general. This can be seen in the following expression accounting for just the first term in (21) which

$$\begin{aligned}\lim_{\eta}\frac{1}{4}\int d\mathbf{p}\,d\mathbf{x}d\mathbf{q}\Phi_1(\mathbf{q})\frac{\eta}{\eta^2+(\mathbf{p}\cdot\mathbf{q}/k_1)^2}\big|\bar{W}(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1})\big|^2\\ = \quad \frac{\pi}{4}\int d\mathbf{p}\,d\mathbf{x}d\mathbf{q}\Phi(\mathbf{q})\delta(\mathbf{p}\cdot\mathbf{q}/k)\big|\bar{W}(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k})\big|^2\end{aligned}$$

which is positive in general. In order to maintain the soundness of (20) the error caused by the regularization $\sqrt{\varepsilon}\eta W_1^\eta$ must be $O(\varepsilon)$. In view of the above calculation (38), it follows that η must be of the same order of magnitude as ε or smaller.

Now, with $\eta=\varepsilon$, the same calculation (38) also implies that the corrector term in (20) does not vanish in the mean-square norm in any dimension, i.e. $\lim_{\varepsilon\to 0}\varepsilon\int\langle|W_1^\varepsilon|^2\rangle\, d\mathbf{x}d\mathbf{p} > 0$ in general.

However, one does have a vanishing corrector in three dimensions in the following slightly weaker sense

$$\lim_{\varepsilon\to 0}\varepsilon\int d\mathbf{x}\,\left\langle\left|\int d\mathbf{p}\,W_1^\varepsilon(\mathbf{x},\frac{\mathbf{x}}{\varepsilon},\mathbf{p})\psi(\mathbf{p})\right|^2\right\rangle=0,\quad d=3 \tag{38}$$

for any smooth, compactly supported function ψ. Indeed, we have from just the first term in (21) that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\frac{\varepsilon}{4}\int d\mathbf{p}d\mathbf{p}'\,d\mathbf{x}d\mathbf{q}\frac{\Phi_1(\mathbf{q})\psi(\mathbf{p})\psi(\mathbf{p}')}{(\varepsilon+i\mathbf{p}\cdot\mathbf{q}/k_1)(\varepsilon-i\mathbf{p}'\cdot\mathbf{q}/k_1)}\bar{W}(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1})\bar{W}^*(\mathbf{x},\mathbf{p}'-\frac{\mathbf{q}}{2k_1})\\
=\;&\lim_{\varepsilon\to 0}\frac{\varepsilon k_1^2}{4}\int d\mathbf{q}\frac{\Phi_1(\mathbf{q})}{|\mathbf{q}|^2}\Big[\pi\int d\mathbf{p}\delta(\mathbf{p}\cdot\hat{\mathbf{q}})\psi(\mathbf{p})\bar{W}(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1})-⨍ d\mathbf{p}\frac{i\psi(\mathbf{p})}{\mathbf{p}\cdot\hat{\mathbf{q}}}\bar{W}(\mathbf{x},\mathbf{p}-\frac{\mathbf{q}}{2k_1})\Big]\\
&\times\Big[\pi\int d\mathbf{p}'\delta(\mathbf{p}'\cdot\hat{\mathbf{q}}/k_1)\psi(\mathbf{p}')\bar{W}(\mathbf{x},\mathbf{p}'-\frac{\mathbf{q}}{2k_1})+⨍ d\mathbf{p}'\frac{i\psi(\mathbf{p}')}{\mathbf{p}'\cdot\hat{\mathbf{q}}/k_1}\bar{W}^*(\mathbf{x},\mathbf{p}'-\frac{\mathbf{q}}{2k_1})\Big]
\end{aligned}$$

where $\hat{\mathbf{q}}=\mathbf{q}/|\mathbf{q}|$. The essential point now is that $|\mathbf{q}|^{-2}$ is an integrable singularity in three dimensions and hence the above expression vanishes in the limit. The similar analysis with W_2 leads to the same conclusion.

In summary, the multi-scale expansion (20) is to be understood in the sense of (38), i.e. strongly in $\mathbf{x}$ but weakly in $\mathbf{p}$ in the mean-square sense. This is physically reasonable as $\mathbf{p}$ corresponds to the scale of fluctuation compatible with that of the refractive index.

Department of Mathematics, University of California, Davis, CA 95616-8633
E-mail address: `fannjiang@math.ucdavis.edu`